2. Exponents

2.1 Exponents: Exponents are shorthand for repeated multiplication of the same thing by itself. For instance, the shorthand for multiplying three copies of the number 5 is shown on the right-hand side of the "equals" sign in $(5)(5)(5) = 5^3$. The "exponent", being 3 in this example, stands for however many times the value is being multiplied. The thing that's being multiplied, being 5 in this example, is called the "base".

The exponent says how many copies of the base are multiplied together. For example, $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$. The base 3 appears 5 times in the repeated multiplication, because the exponent is 5. Here, 3 is the *base*, 5 is the *exponent*, and 243 is the *power* or, more specifically, *the fifth power of 3, 3 raised to the fifth power*, or *3 to the power of 5*.

The word "raised" is usually omitted, and very often "power" as well, so 3^5 is typically pronounced "three to the fifth" or "three to the five".

The exponent is usually shown as a superscript (a letter, character, or symbol that is written above) to the right of the base. The exponentiation b^n can be read as: ***b** raised to the **n**-th power*, ***b** raised to the power of **n***, or ***b** raised by the exponent of **n***, most briefly as ***b** to the **n***. superscript

This process of using exponents is called "raising to a power", where the exponent is the "power". The expression "5^3" is pronounced as "five, raised to the third power" or "five to the third". There are two specially-named powers: "to the second power" is generally pronounced as "squared", and "to the third power" is generally pronounced as "cubed". So "5^3" is commonly pronounced as "five cubed".

The expression $b^2 = b \cdot b$ is called the square of b because the area of a square with side-length b is b^2. It is pronounced "b squared". The expression $b^3 = b \cdot b \cdot b$ is called the cube of b because the volume of a cube with side-length b is b^3. It is pronounced "b cubed".

When we deal with numbers, we usually just simplify; we'd rather deal with "27" than with "3^3". But with variables, we need the exponents, because we'd rather deal with "x^6" than with "*xxxxxx*".

The power b^n can be defined also when n is a negative integer, for nonzero b. No natural extension to all real b and n exists, but when the base b is a positive real number, b^n can be defined for all real and even complex exponents n via the exponential function e^z. Trigonometric functions can be expressed in terms of complex exponentiation.

Exponentiation where the exponent is a matrix is used for solving systems of linear differential equations. Exponentiation is used pervasively in many other fields, including economics, biology, chemistry, physics, as well as computer science, with applications such as compound interest, population growth, chemical reaction kinetics, wave behavior and public-key cryptography.

Exponentiation may be generalized from integer exponents to more general types of numbers.

2.2 Exponent Laws

One of the trickiest concepts in algebra involves the manipulation of exponents, or powers. Many times, problems will require you to simplify variables with exponents, or you will have to simplify an equation with exponents to solve it. To work with exponents, you need to know the basic exponent laws.

Adding and Subtracting with Non-like Terms

When a problem gives you two terms, or chunks, that do not have the exact same variables, or letters, raised to the exact same exponents, you cannot combine them.

Adding Like Terms

If two terms have the same variables raised to the exact same exponents, add their coefficients and use the answer as the new coefficient for the combined term. The exponents remain the same. For instance, 3x^2 + 5x^2 would turn into 8x^2.

Subtracting Like Terms

If two terms have the same variables raised to the exact same exponents, subtract the second coefficient from the first and use the answer as the new coefficient for the combined term. The powers themselves do not change. For example, 5y^3 - 7y^3 would simplify to -2y^3.

Multiplying

When multiplying two terms (it does not matter if they are like terms), multiply the coefficients together to get the new coefficient. Then, one at a time, add the powers of each variable to make the new powers.

Two Powers of the Same Base

Suppose you have $(x^5)(x^6)$; how do you simplify that? Just remember that you're counting factors.

$$x^5 = (x)(x)(x)(x)(x) \text{ and } x^6 = (x)(x)(x)(x)(x)(x)$$

Now multiply them together:

$$(x^5)(x^6) = (x)(x)(x)(x)(x)(x)(x)(x)(x)(x)(x) = x^{11}$$

Why x^{11}? Well, how many x's are there? Five x factors from x^5, and six x factors from x^6, makes 11 x factors total. Can you see that whenever you multiply *any* two powers of the same base, you end up with a number of factors equal to the total of the two powers? In other words, **when the bases are the same**, you find the new power by just **adding the exponents**:

Powers of Different Bases

Caution! The rule above works only when multiplying powers of the same base. For instance,

$$(x^3)(y^4) = (x)(x)(x)(y)(y)(y)(y)$$

If you write out the powers, you see there's no way you can combine them. Except in one case: **If the bases are different but the exponents are the same**, then you can **combine them**. Example:

$$(x^3)(y^3) = (x)(x)(x)(y)(y)(y)$$

But you know that it doesn't matter what order you do your multiplications in or how you group them. Therefore,

$$(x)(x)(x)(y)(y)(y) = (x)(y)(x)(y)(x)(y) = (xy)(xy)(xy)$$

But from the very definition of powers, you know that's the same as $(xy)^3$. And it works for any common power of two different bases:

It should go without saying, but I'll say it anyway: all the laws of exponents work in both directions. If you see $(4x)^3$ you can decompose it to $(4^3)(x^3)$, and if you see $(4^3)(x^3)$ you can combine it as $(4x)^3$.

First Power Exponent Rule

Anything raised to the first power stays the same. For example, 7^1 would just be 7.

Exponents of Zero

Anything raised to the power of 0 becomes the number 1. It does matter how complicated or big the term is.

Dividing Powers

What about dividing? Remember that dividing is just multiplying by 1-over-something. So all the laws of division are really just laws of multiplication.

Example: What is $x^8 \div x^6$? Using the definition of negative exponents that's just $x^8(x^{-6})$. Now use the product rule (two powers of the same base) to rewrite it as x^{8-6}, or x^2. However you slice it, you come to the same answer: **for division with like bases you subtract exponents**, just as for multiplication of like bases you add exponents:

But there's no need to memorize a special rule for division: you can always work it out from the other rules or by counting.

In the same way, **dividing different bases** can't be simplified unless the exponents are equal. $x^3 \div y^2$ can't be combined because it's just xxx/yy; But $x^3 \div y^3$ is xxx/yyy, which is (x/y)(x/y)(x/y), which is $(x/y)^3$.

Negative Exponents

A negative exponent means to **divide** by that number of factors **instead of multiplying**. So 4^{-3} is the same as $1/(4^3)$, and $x^{-3} = 1/x^3$.

As you know, you can't divide by zero. So there's a restriction that $x^{-n} = 1/x^n$ only when x is not zero. When $x = 0$, x^{-n} is undefined.

2.3 Logarithms

The logarithm of a number is the exponent to which another fixed value, the base, must be raised to produce that number. For example, the logarithm of 1000 to base 10 is 3, because 10 to the power 3 is 1000: $1000 = 10 \times 10 \times 10 = 10^3$. More generally, for any two real numbers *b* and *x* where *b* is positive and $b \neq 1$. The logarithm to base 10 ($b = 10$) is called the common logarithm and has many applications in science and engineering.

The natural logarithm has the irrational (transcendental) number *e* as its base; its use is widespread in pure mathematics, especially calculus. The binary logarithm uses base 2 ($b = 2$) and is prominent in computer science.

Logarithms were introduced by John Napier in the early 17th century as a means to simplify calculations. They were rapidly adopted by navigators, scientists, engineers, and others to perform computations more easily, using slide rules and logarithm tables. Tedious multi-digit multiplication steps can be replaced by table look-ups and simpler addition because of the fact—important in its own right—that the logarithm of a product is the sum of the logarithms of the factors.

Logarithms are the "opposite" of exponentials, just as subtraction is the opposite of addition and division is the opposite of multiplication. Logarithms "undo" exponentials. Technically speaking, logs are the inverses of exponentials. WHEN we are given the base 2, for example, and exponent 3, then we can evaluate 2^3.

$$2^3 = 8.$$

Inversely, if we are given the base 2 and its power 8 --

$$2^? = 8$$

-- then what is the exponent that will produce 8?

That exponent is called a logarithm. We call the exponent 3 the logarithm of 8 with base 2. We write

$$3 = \log_2 8.$$

The base 2 is written as a subscript.

3 is the *exponent* to which 2 must be raised to produce 8.

A logarithm is an exponent.

Since

$$10^4 = 10{,}000$$

then

$$\log_{10} 10{,}000 = 4.$$

"The logarithm of 10,000 with base 10 is 4."

4 is the *exponent* to which 10 must be raised to produce 10,000.

"$10^4 = 10{,}000$" is called the exponential form.

"$\log_{10} 10{,}000 = 4$" is called the logarithmic form.

Here is the definition:

$$\log_b x = n \quad \text{means} \quad b^n = x.$$

That base with that exponent produces x